Lecture1: Mathematical Model of Electrical Systems

1. Introduction to Control Systems:

A control system is a system of devices or set of devices, that manages, commands, directs, or regulates the behavior of other devices or systems using control loops to achieve desired results. It can range from a single home heating controller using a thermostat controlling a domestic boiler to large Industrial control systems which are used for controlling processes or machines.

The following figure shows the simple block diagram of a control system. Here, the control system is represented by a single block. Since, the output is controlled by varying input, the control system got this name.

2. Mathematical Models of Control System:

The control systems can be represented with a set of mathematical equations known as **mathematical model**. These models are useful for analysis and design of control systems. Analysis of control system means finding the output when we know the input and mathematical model. Design of control system means finding the mathematical model when we know the input and the output.

The following mathematical models are mostly used:

- Differential equation model
- Transfer function model
- State space model

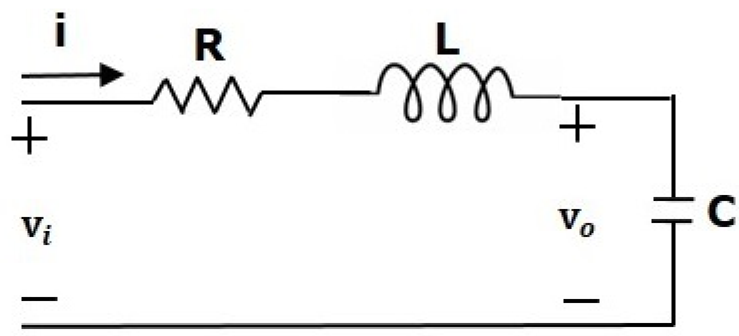
Let us discuss the first two models in this lecture.

2.1 Differential Equation Model:

Differential equation model is a time domain mathematical model of control systems. Follow these steps for differential equation model:

- Apply basic laws to the given control system.
- Get the differential equation in terms of input and output by eliminating the intermediate variable(s).

Example: Consider the following electrical system as shown in the following figure. This circuit consists of resistor, inductor and capacitor. All these electrical elements are connected in **series**. The input voltage applied to this circuit is v_i and the voltage across the capacitor is the output voltage v_o.



Writing the differential equations with the help of Kirchhoff's voltage law:

$$v_i = Ri + L\frac{di}{dt} + v_o$$

Substitute, the current passing through capacitor in $i = c\frac{dv_o}{dt}$ the above equation.

$$\Rightarrow v_i = RC\frac{dv_o}{dt} + LC\frac{d^2v_o}{dt^2} + v_o$$

$$\Rightarrow \frac{d^2v_o}{dt^2} + \left(\frac{R}{L}\right)\frac{dv_o}{dt} + \left(\frac{1}{LC}\right)v_o = \left(\frac{1}{LC}\right)v_i$$

The above equation is a second order **differential equation**.

2.2 Transfer Function Model:

The transfer function of a system is the ratio of Laplace transforms of output and input quantities, initial conditions being zero.

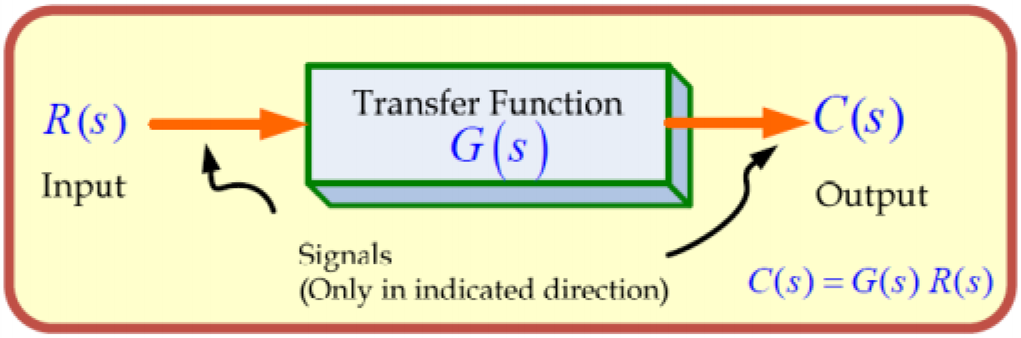


Figure (): Single block diagram representation.

If we have an input function of R(s), and an output function of C(s), we define the transfer function G(s):

$$G(s) = \frac{C(s)}{R(s)}$$

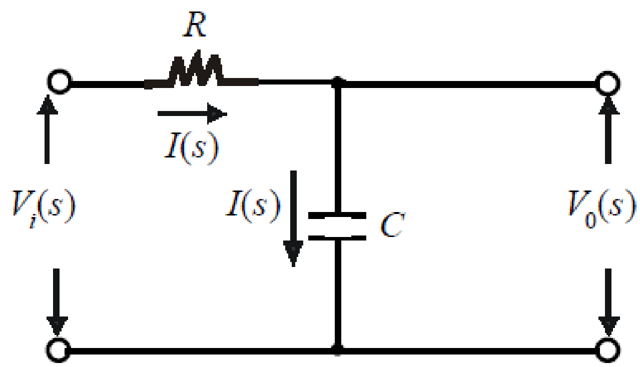
When a physical system is analyzed, a mathematical model is prepared by writing differential equations with help of various laws. The steps involved in obtaining the transfer function are:

- Write differential equation of the system.
- Replace terms involving $\frac{d}{dt}$ by s and $\int dt$ by $\frac{1}{s}$.
- Eliminate all but the desired variable.

3. Mathematical Model of Electrical Systems:

Most of the electrical systems can be modelled by three basic elements: Resistor, inductor, and capacitor. Circuits consisting of these three elements are analysed by using Kirchhoff's Voltage law and Current law.

Example 1: Derive the transfer function of the circuit shown in Figure below:



Solution:

First Step: Writing the differential equations with the help of Kirchhoff's voltage law:

$v_{i(t)} = Ri_{(t)} + v_{o(t)}$; and

$$v_{o(t)} = \frac{1}{C}\int i\, dt$$

Second Step: Taking Laplace transform, we get:

$V_{i(s)} = R\, I(s) + V_o(s)$;

Third Step: Eliminate all but the desired variable, we get:

and $V_o(s) = \frac{1}{sC} I(s)$

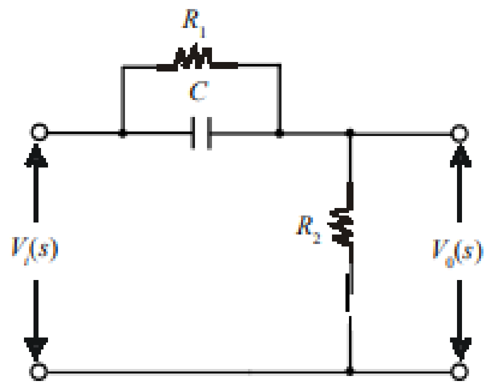
or $I(s) = sC\, V_o(s)$

or $V_{i(s)} = s\, RC\, V_o(s) + V_o(s)$

$V_{i(s)} = V_o(s)(1 + s\, RC)$

or $\frac{V_o(s)}{V_i(s)} = \frac{1}{1+s\, RC}$

Example 2: Find the transfer function of the electrical network shown in Figure below. If $v_1 = 8\, sin10t\ volts,\ R_1 = 50\ K\ ohms, R_2 = 5\ K\ ohms$ and $C = 1\ \mu F$. Then, calculate the output voltage in magnitude and in phase relative to input voltage.



Solution:

If Z_1 is the equivalent impedance of the parallel combination of R_1 and C then:

$$Z_1 = R_1 \| X_C = \frac{R_1\, X_C}{R_1 + X_C}\ ; where\ \ X_C = \frac{1}{j\omega C}$$

$$Z_1 = \frac{R_1 \frac{1}{j\omega C}}{R_1 + \frac{1}{j\omega C}} = \frac{R_1}{1 + jR_1\omega C}; where\ j\omega = s$$

$$Z_1 = \frac{R_1}{1 + sCR_1}$$

Writing the differential equations by Kirchhoff's voltage law, we get:

$v_{i(t)} = Z_1 i(t) + v_o(t)$; and

$v_o(t) = R_2 i(t)$

Taking Laplace transform, we get:

$V_{i(s)} = Z_1\, I(s) + R_2 I(s)$; and

$V_o(s) = R_2 I(s) \qquad\qquad \therefore I(s) = \frac{V_o(s)}{R_2}$

or $V_{i(s)} = Z_1\ \frac{V_o(s)}{R_2} + V_o(s)$

or $V_{i(s)} = V_o(s)(\frac{Z_1}{R_2} + 1)$

substituting the value of Z_1, we get:

$$V_{i(s)} = V_o(s)(\frac{R_1}{1 + sCR_1}\ \frac{1}{R_2} + 1)$$

$$G(s) = \frac{V_o(s)}{V_{i(s)}} = \frac{R_2(1 + sCR_1)}{R_1 + R_2 + sCR_1R_2}$$

$$\therefore G(s) = \frac{5000(1 + s \times 1 \times 10^{-6} \times 50 \times 1000)}{50 \times 1000 + 5000 + s \times 1 \times 10^{-6} \times 50 \times 1000 \times 5000}$$

$$= \frac{0.1(1 + 0.05\, s)}{1 + 0.0045\, s}$$

Put s=$j\omega$. Therefore**:**

$$G_{j\omega} = \frac{0.1(1 + 0.05\ \ j\omega)}{1 + 0.0045\ \ j\omega}$$

$$\angle G_{j\omega} = tan^{-1}0.05\,\omega - tan^{-1}0.0045\,\omega$$

Since $v_1 = 8\,sin10t$, therefore $\omega = 10$

$$\therefore \angle G_{j\omega} = tan^{-1}0.05 \times 10 - tan^{-1}0.0045\ \times 10 = 26.265^o - 2.576^o = 23.989^o$$

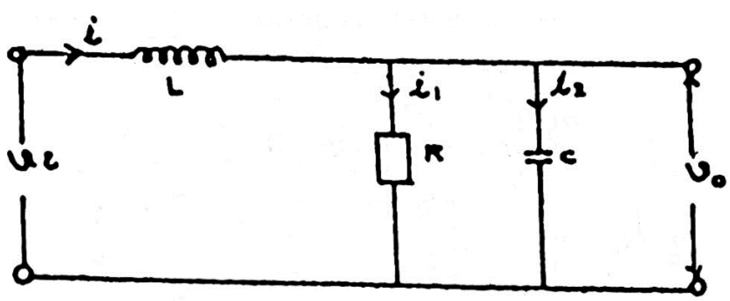
Also, $M = |G_{j\omega}| = \frac{0.1\sqrt{1+(0.05\omega)^2}}{\sqrt{1+(0.045\omega)^2}}$

Putting $\omega = 10$, we get:

$$M = \frac{0.1\sqrt{1 + (0.5)^2}}{\sqrt{1 + (0.45)^2}} = 0.1019$$

Therefore $V_o = 8 \times 0.1019 = 0.8156\,Volts$

Example 3: Find the transfer function of the electrical network shown in Figure below



Solution:

Assuming no external load: $i = i_1 + i_2$

Applying Kirchhoff's voltage law to the electrical network, we get:

$$v_{i(t)} = \frac{L\,d\,i(t)}{dt} + R\,i_1(t)$$

And $v_{o(t)} = R\,i_1(t) = \frac{1}{C}\int i_2(t)\,dt$

Taking Laplace transform, we get:

$V_i(s) = sL\,I(s) + V_o(s)$ and

$$V_o(s) = \frac{I_2(s)}{sC} = R\,I_1(s)$$

or $V_i(s) = V_o(s) + sL(I_1(s) + I_2(s))$

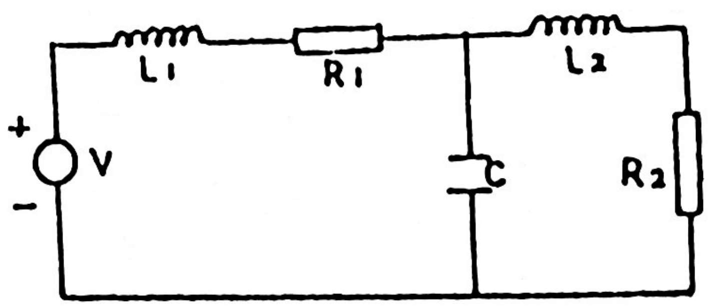
or $V_i(s) = V_o(s) + sL(\frac{V_o(s)}{R} + sC\,V_o(s))$

or $V_i(s) = V_o(s)\left[1 + \frac{sL}{R} + s^2LC\right]$

or $\frac{V_o(s)}{V_i(s)} = \frac{1}{s^2LC + \frac{sL}{R} + 1}$

Homework:

1. Write the differential equations for the electrical network shown in Figure below:



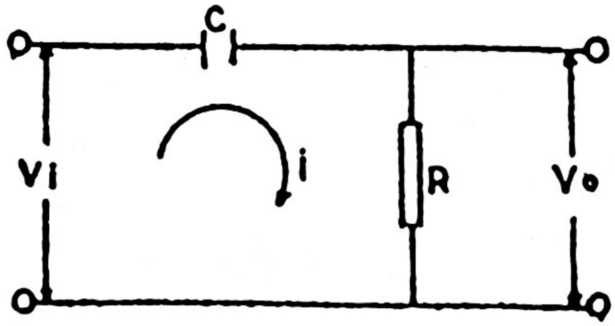
Ans.

$$\boldsymbol{v_{i(t)} = \frac{L_1\,d\,i_1(t)}{dt} + R_1\,i_1(t) + \frac{1}{C}\int i_1(t)\,dt - \frac{1}{C}\int i_2(t)\,dt}$$

And

$$\boldsymbol{0 = \frac{L_2\,d\,i_2(t)}{dt} + R_2\,i_2(t) + \frac{1}{C}\int i_2(t)\,dt - \frac{1}{C}\int i_1(t)\,dt}$$

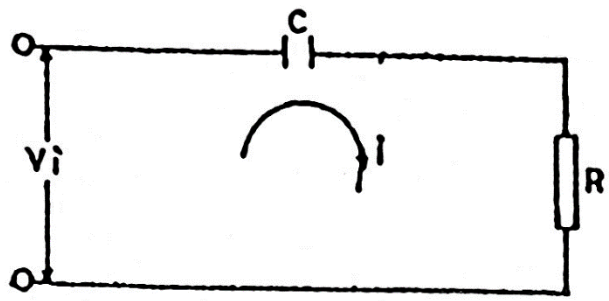
2. Derive the transfer function of the circuit shown in Figure below:



Ans.

$$\frac{V_o(s)}{V_i(s)} = \frac{sRC}{1 + sRC}$$

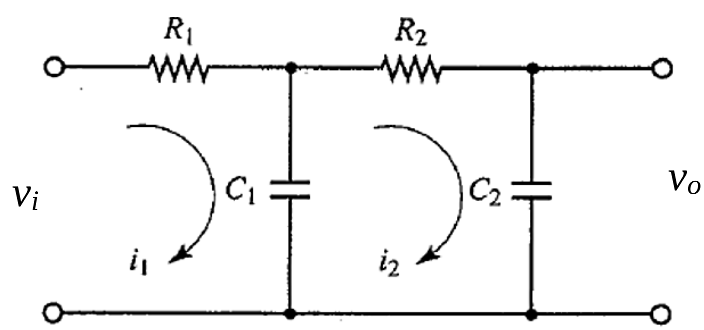
3. Derive the transfer function of the circuit shown in Figure below:



Ans.

$$\frac{I(s)}{V_i(s)} = \frac{1}{R + \frac{1}{sC}} = \frac{sC}{1 + s\,CR}$$

4. Find the transfer function of the electrical network shown in Figure below.



Ans.

$$\frac{V_o(s)}{V_i(s)} = \left[\frac{\frac{1}{C_1 s}\,\frac{1}{C_2 s}}{R_1\frac{1}{C_1 s} + R_1R_2 + R_1\frac{1}{C_2 s} + \frac{1}{C_1 s}R_2 + \frac{1}{C_1 s}\frac{1}{C_2 s}}\right]$$

$$OR \quad = \frac{1}{1 + s(R_1C_1 + R_2C_2 + R_1C_2) + s^2C_1C_2R_1R_2}$$